Dependency Categories

Status: DRAFT –TREAT WITH CAUTION

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Ad Otium

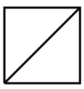
August 7, 2018

Under a new name, *dependency category*, this paper corrects the description of *network categories* given in [2]. Dependency categories are a variation on the notion of contextual category ([1], [3]) in that the category of dependency categories, as defined here, is equivalent to the category of contextual categories.

As in [2] the motivation has been to formalise and foreground the network structure of type dependencies so as to provide a theoretical foundation for the use of networks of composition relationships in entity modelling as applied both conceptually and, for the most practical of purposes, in systems development.

The objects of categories of either persuasion, either contextual categories or dependency categories, can be thought of as contexts or as types that vary (also known as dependent types) but equally they can be thought of as entity types in the sense used in entity modelling. The morphisms of either correspond with the many-one binary relationships that are the staple fare of entity modelling and in either case there is a distinguished subset of morphisms depicted on diagrams using a triangular headed arrow (—▹). In a contextual category the distinguished subset forms a hierarchy, in a dependency category the distinguished morphisms form not a hierarchy but a wide acyclic subcategory. In either case the distinguished morphisms correspond to certain relationships known as composition relationships which feature in certain styles of entity modelling ([6], [9], [4]) and which were implied in the influential paper by Chen [5] who introduced the idea of certain entities being dependent on binary relationships with others for both their identification and their existence.

1 Introduction

In this figure  there are two triangles and these have six sides – though the lines depicting these number only five; this possibility comes about because in the statement that the two triangles have six sides we are understanding *side* to be *side of triangle* which is to say that we understand it to be a concept that varies as triangle varies. A side, therefore, is a dependent type of thing – it is some thing held in the mind in the context of some other thing. Similarly, certain words, such as the word ‘pop’, appear in sentences sometimes as nouns but also sometimes as verbs, or adjectives or as adverbs, so illustrating that a word, in and by itself, cannot be said to be a ‘noun’, ‘verb’, ’adjective’ or other such part of speech, lest it be appearing in some grammatical sentence and so ‘noun’ and ‘verb‘ and the like – as types of thing – are dependent on sentence-like types of thing.

If in a situation temperature varies over position then it is implied - and might educate us if our notion of temperature was only half formed - that the concept of temperature had somehow in its make up a dependency on the concept of spatial position. So it is with types that vary. Concepts like ‘angle’, ‘edge’, ‘boundary’, ‘bounding line’ as we learn these concepts, we learn that they do not stand alone; they are

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dependent concepts as are face of cube, endpoint of line, junction between lines, citizen (of a country), tangent (to a curve), atom of a molecule, nucleus of a cell, character of a play.

When a quantity y varies as some variable x varies then as matter of course the quantity might be written y_x with the x subscript reminding us of the dependency; or the quantity might be written $y(x)$ so using the notation of function application. If the quantities in question are real numbers then we may write either

$$y_{x) x \in R}$$

or as something along the lines of:

$$\text{for } x \in R, y(x) \in R$$

Equally we might introduce y at the outset not as a quantity but as a function y, $y : R \to R$. Note however that in a particular scientific enquiry, at the outset, what comes first is the quantities that are measured and the enquiry will often be to understand the quantities which can be measured and to understand over what domains they vary influenced by which variables - to discover the independent and dependent variables. The point that is being made here is that in scientific enquiry *quantities that vary* precede functions and quantities that vary do so according to context.

In a formal mathematical notation ([3] we introduce symbols for functions and, subsequently, dependent types by way of formal rules. For example we introduce types A and B and some function f delivering entities of type B from entities of type A by the rules:

$$\vdash A \text{ is a type} \tag{1}$$
$$\vdash B \text{ is a type} \tag{2}$$
$$x \in A \vdash f(x) \in B \tag{3}$$

In rules such as these, variables are introduced to the left of the turnstyle($\vdash$) and an assertion is given to the right. The left hand side presents a context in which the right hand side is asserted. We also write $f : A \to B$ for such a function and consider there to be a category of types and functional dependencies and represent such in the diagrams of catgeory theory.

A somewhat different diagrammatic notation is available within the sphere of Information Systems Development, entity model diagrams in the style of Barker, Ellis, Martin, Schlaer-Mellor and others (see [10]), focus on binary relationships between types of entity – functional dependencies between types of entity are said to be many-one relationships and are depicted using the crows foot notation as, for example, here:

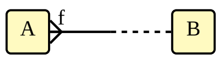


Figure 1: Many-one Binary Relationship f between entity types A and B

Of course there are quantities which do not vary – they are constant – they do not have contexts they depend on. If a is a constant of type A then we assert:

$$\vdash a \in A$$

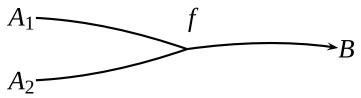
Using the symbol R for the type of real numbers then we have:

$$\vdash R \text{ is a type}$$
$$\vdash \pi \in R$$

Other quantities vary over a number of different dimensions and we represent them as functions with multiple arguments. If a quantity of type f varies over types A_1 and A_2, as for example electric potential varies over space and time then we write it as a function with two arguments:

$$\vdash A_1 \text{ is a type}$$
$$\vdash A_2 \text{ is a type}$$
$$\vdash B \text{ is a type}$$
$$x_1 \in A_1, x_2 \in A_2 \vdash f(x_1, x_2) \in B$$

We might also represent f by a multiarrow in a graph:



or as a function whose domain is the cartesian product of $A1$ and $A2$, which generally would be represented as $A_1 \times A_2$.

Consider that types are a kind of abstract quantity and we are led to the possibility that types may vary just as quantities do and in particular that the *type* of a quantity may vary as the *quantity* itself does.

Various notations are used to express such type dependencies. For formal mathematical purposes, the functional notation used for quantities that vary can equally be used for types that vary. If the type of quantity f varies as f varies then we can assert this as follows:

$$\vdash A \text{ is a type}$$
$$x \in A \vdash B(x) \text{ is a type}$$
$$x \in A \vdash f(x) \in B(x)$$

Such a B is said to be dependent on type A. So, for example, 'the side opposite an angle' in a study of triangles is a quantity (an entity) which varies both as the triangle varies and as the angle varies:

$$x \in Triangle, y \in Angle(x) : oppositeSide(y) \in Side(x)$$

Similarly we can make type assertions such as

$$x \in Country \vdash headOfState(x) \in Citizen(x)$$

In this type assertion 'headOfState' is a quantity which is varying, dependent on Country and is of a type which varies as the country varies.

If B is a type dependent on A then instead of imagining type B as having a set of instances we can imagine instead an A-indexed family of sets of instances. If in addition there is a b such that $x \in A, b(x) \in B(x)$ (as 'headOfState' above) then think of a section of the family of sets of instances of B, see figure 2. If $Ba)a \in A$ in an A-indexed family of sets the a section of B is a function $b : A \to \bigcup_{a \in A} B(a)$ such that for each $a \in A, b(a) \in B(a)$.
A second possibility, by analogy with the representation of a function as an arrow $f : B \to A$, is to represent a dependency between one type B and another A by a directed edge $B \rightarrowtriangle A$ which then represents the functional relationship between entities of type B and the entities of type A that they depend on.

In this case in any particular situation the types and their dependencies form a directed graph.

Furthermore, any directed graph makes sense as, and can be interpreted as, a set of types and type dependencies providing (i) there are no cycles in the graph and provided that (ii) all nodes B there are only finitely many A, such that $B \rightarrowtriangle A$ and (iii) there are no infinite sequences of the form $A_1 \rightarrowtriangle A_2 \rightarrowtriangle A_3 \ldots$..

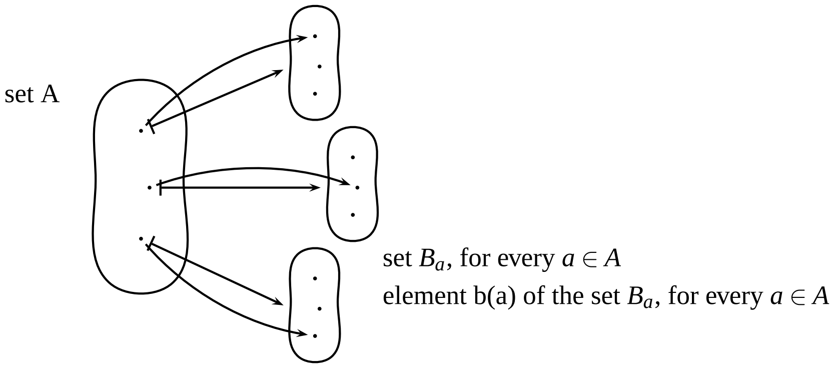
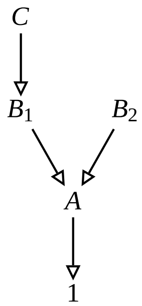


Figure 2: A section b of A-indexed family of sets B.

As an example, the directed graph:

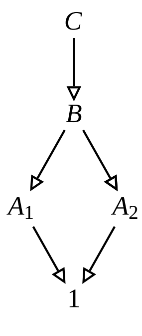


(4)

interpreted as types and type dependencies expresses the following:

$$A \text{ is a type} \tag{4a}$$
$$x \in A : B_1(x) \text{ is a type} \tag{4b}$$
$$x \in A : B_2(x) \text{ is a type} \tag{4c}$$
$$x \in A, y \in B_1(x) : C(x,y) \text{ is a type} \tag{4d}$$

Similarly the directed graph:



(5)

can be interpreted as representing the following type system:

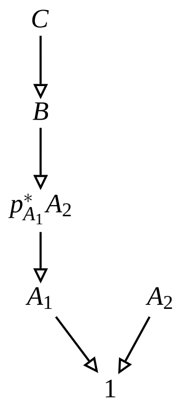
A_1 is a type (5a)

A_2 is a type (5b)

$x_1 \in A_1, x_2 \in A_2 : B(x_1, x_2)$ is a type (5c)

$x_1 \in A_1, x_2 \in A_2, y \in B(x_1, x_2) : C(x_1, x_2, y)$ is a type (5d)

and this example can be represented in form (6) within a contextual category werein the cartesian product of A_1 and A_2 can represented as $p^*_{A_1}A_2$ In this example type B is explicitly dependent on two types (A_1 and A_2). In the approach of [1], [3] this double dependency needs be represented (in a contextual category) by a dependency of B on a cartesian product of A_1 and A_2. Two such cartesian products are available: $p^*_{A_1}A_2$ and $p^*_{A_2}A_1$. The resulting graph if the first one is chosen is shown in (6).



(6)

Such type dependencies as these can be represented by *composition relationships* within certain styles of entity modelling (for one of these, see www.entitymodelling.org). Composition relationships may be distinguished on diagrams from other functional relationships by being drawn leaving the lower edge of a box representing the type being depended on and entering the upper edge of the box representing the dependent type.

Examples are given in figures 3 and **??**.

	Symbol	*Introductory Rule*
	language	$\vdash \mathit{language}$ is a type
	sentence	$x \in \mathit{language} \vdash \mathit{sentence}(x)$ is a type
(a)	word	$x \in \mathit{language} \vdash \mathit{word}(x)$ is a type
	noun	$x \in \mathit{language},\ y \in \mathit{sentence}(x) \vdash \mathit{noun}(y)$ is a type
	verb	$x \in \mathit{language},\ y \in \mathit{sentence}(x) \vdash \mathit{verb}(y)$ is a type
	adjective	$x \in \mathit{language},\ y \in \mathit{sentence}(x) \vdash \mathit{adjective}(y)$ is a type

(b)

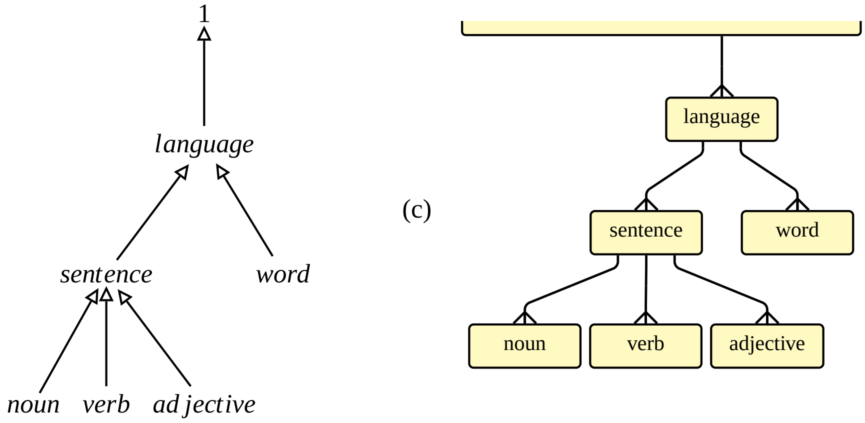


Figure 3: Three representations of a system of types (a) rules in a formal mathematical syntax, (b) a graph of type dependencies (c) an entity modelling diagram.

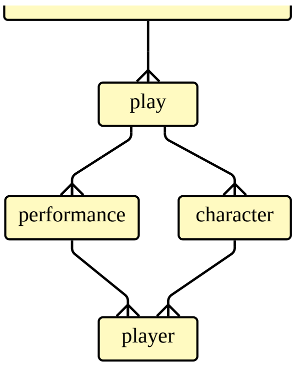


Figure 4: A player plays the part of a character in the same play that they are performing in.

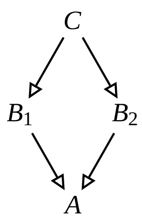
Consider the following two rules:

$$x_1 \in A, y_1 \in B_1(x_1), x_2 \in A, y_2 \in B_2(x_2) : C(x_1, y_1, x_2, y_2) \text{ is a type} \tag{7a}$$

and

$$x \in A, y_1 \in B_1(x), y_2 \in B_2(x) : C(x, y_1, y_2) \text{ is a type} \tag{7b}$$

They both give rise to a graph shaped like this:



(8)

In the case of (7b), C relies for context on instances of B_1 and B_2 which themselves are based within the context of a common instance of A. In the case of (7b), but not, significantly, in the case (7a), the path: $C \to B_1 \to A$ is equivalent to the path: $C \to B_2 \to A$ and the diagram commutes. So dependencies compose in different ways and so represent the types dependencies (the variables over which types vary); therefore it is the category of dependencies (not just the directed graph) which is significant to characterising the type dependency relationships. Note that the model in figure 4 follows the pattern of (7b) rather than (7a) which is to say that the diagram of relationships in figure 4 commutes.

We now proceed to give a series of definitions which lead eventually to the definition of *dependency category*. Dependency categories contain all the structure needed to represent dependent types in the sense that these are in the syntax of generalised algebraic theories ([3]). As such the category of dependency categories is equivalent to the category of contextual categories.

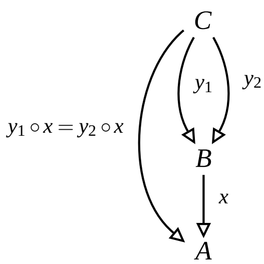
2 Acyclic Categories

A category C is called *acyclic*, if it has no inverses and no nonidentity endomorphisms. This definition is given by Kozlov (see [8]) who offers the following intuition:

> *Another way to visualize acyclic categories is to think of them as those that can be drawn on a sheet of paper, with dots indicating the objects, and straight or slightly bent arrows, all pointing down, indicating the nonidentity morphisms...*

Previous authors had referred to such categories as being *loop-free*.

The following is an example of an acyclic category given by Kolozov which we have rearranged and relabelled:



(9)

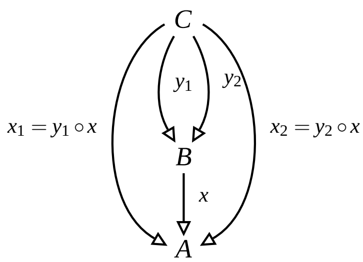
This category can be taken as representing the following type system:

$$A \text{ is a type} \tag{8a}$$

$$x \in A : B(x) \text{ is a type} \tag{8b}$$

$$x \in A_1, y_1 \in B(x), y_2 \in B(x) : C(x, y_1, y_2) \text{ is a type} \tag{8c}$$

Contrast with this acyclic category:



(9)

which represents :

$$A \text{ is a type} \tag{8a}$$
$$x \in A : B(x) \text{ is a type} \tag{8b}$$
$$x_1 \in A_1, y_1 \in B(x_1), x_2 \in A_1, y_2 \in B(x_2) : C(x_1, x_2, y_1, y_2) \text{ is a type} \tag{8c}$$

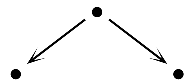
3 Category with Distinguished Morphisms

Definition A category **C** is *well-founded* provided that for any object A of C the set of morphisms with domain A is finite.

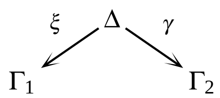
Definition Define a *category with distinguished morphisms* to be a category **C** along with a wide acyclic subcategory **D** that is well-founded. Morphisms of the subcategory are referred to as d-morphism. We will write $f : B \rightarrowtriangle A$ in **C** to mean that $f : B \to A$ in **C** and that f is a d-morphism i.e is in the subcategory **D** of distinguished morphisms[1].

4 Spans and Cospans

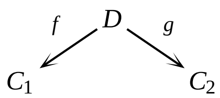
The category Λ is defined to be this category:



If we need names for the individual objects and morphisms we will use the names show here:

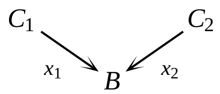


A *span* within a category **C** is exactly a functor $S : \Lambda \to \mathbf{C}$; it is a pair of morphisms of **C** that have a common domain object i.e it is any diagram of this form:



(9)

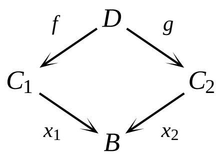
Similarly a *cospan* in a category **C** is exactly a functor $S : \Lambda^{op} \to \mathbf{C}$ and so it is exactly a diagram of this form:



(10)

[1] Note that this the same notation has been used differently in [3] – a contextual category does give rise to a category with distinguished morphisms but only if the distinguished morphisms are taken to be not just those denoted $p_B : B \rightarrowtriangle A$ in that paper but also all those in the subcategory generated by such.

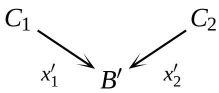
The cospan (10) of C_1 and C_2 is said to be a *coincident cospan* of the span (9) iff the diagram:



(11)

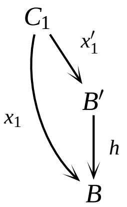
commutes.

We will say that the cospan (10) is a *minimal coincident cospan* for the span (9) iff it is coincident and there does no exist a cospan

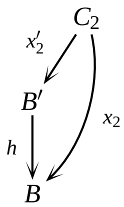


(12)

that is coincident to (9) and such that there is a morphism $h : B' \to B$ such that both



and

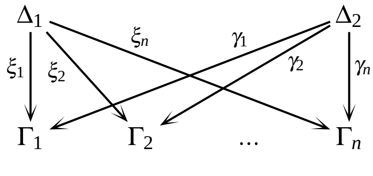


commute. A cospan $F : \Lambda^{op} \to \mathbf{C}$ is said to *factor through* a diagram $J : S \to \mathbf{C}$ iff there exists a cospan $F' : \Lambda^{op} \to S$ such that $F' \circ J = F$.

5 Higher Cospans and Quotiented Higher Cospans

The notion of a *higher cospan* was introduced in [7] for diagrams in a category of shape Λ^{op^n}but for us there are significant diagrams whose shape category is a quotient of Λ^{opn} and since we are not aware of further terminology in this area we introduce some here. It is appropriate to make use the join symbol ($\bowtie$) from relational algebra.

For any $n \geq 2$ define the category $\bowtie_n$to be this category:



Definition A *quotiented n-cospan* in category **C** is any diagram in **C** having shape $\bowtie_n$ i.e. it is any function $F : \bowtie_n \to \mathbf{C}$

6 Some Other Preliminary Definitions[2]

Definition If **C** is a category then a *characterising family* for a span s of **C** is defined to be the set of its minimal coincident cospans.

Definition If **C** is a category with distinguished morphisms and if $\langle f_1, \dots f_n \rangle$ is a tuple of d-morphisms of **C** with common domain then define the *characterising diagram* of $\langle f_1, \dots f_n \rangle$ to be the diagram with shape category $\bowtie_n$where

$$n = \sum_{\substack{i, \\ 1 \le i \le n}} \sum_{\substack{j, \\ 1 \le j \le n, \\ j \ne i}} |\chi_{i,j}|$$

and where $\chi_{i,j}$ is the characterising family within the subcategory of d-morphisms of **C** for the pair f_i, f_j and $|\chi_{i,j}|$ is its cardinality, and with functor $D : S \to \mathbf{C}$ defined for $1 \le i \le n$, $1 \le j \le n$, $i \ne j$, $1 \le k \le |\chi_{i,j}$ by

$$\xi_{i,j,k} \mapsto F_{i,j,k}(\xi)$$

and

$$\gamma_{i,j,k} \mapsto F_{i,j,k}(\gamma)$$

where $F_{i,j,k}$ is the k'th cospan within the characteristic family of the pair f_i, f_j.

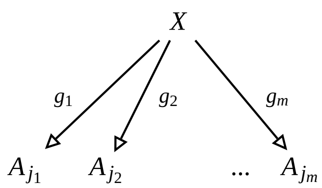
Definition If **C** is a category with distinguished morphisms then a family of d-morphisms $f_i : B \twoheadrightarrow A_i$ is said to be a basis for object B iff for every morphism $f : B \twoheadrightarrow X$ in **C** either

(i) f factors through one of f_i for some i i.e there exists $g : A_i \to X$, for some i, such that $f_i \circ g = f$ or

(ii) there exists $1 \le k_1 \le k_2 \dots k_m \le n$, and d-morphisms $g_1, \dots, g_m$ such that for each j, $1 \le j \le m$, $g_j : X \twoheadrightarrow A_{k_j}$ and

$$f \circ g_j = f_{k_j}$$

and such that the diagram



is a limit cone[3] in category **C** for the characteristic diagram of $g_1, \dots g_m$.

7 Cones and Tight Cones

Recall the definition of cone $\langle N, \psi \rangle$ to a diagram $F : J \to C$ of a category C as an object N of C and a family of morphisms ψ_j indexed by objects j of J such that for all $f : j \to j'$ in J, $\psi_j \circ F(f) = \psi_{j'}$. Define a cone $\langle N, \psi \rangle$to be *tight* iff for all pairs j_1, j_2 of objects of shape category J, every minimal coincident cospan of the span:

[2]Please regard the terminology introduced here as provisional – the author would be glad to receive suggested improvements especially where he has failed to use prior established terminology.

[3]I want to say cannonical limit here but I can't. Monitor this situation to see whether a problem or not. I think it is a minor annoyance. The question is whether if X dependent on Y and Y' isomorphic to Y then X is dependent on Y'. This in turn has an effect on the round trip from presentation to category back to presentation.

$$\begin{array}{ccc} & N & \\ {}^{\psi_{j_1}}\swarrow & & \searrow^{\psi_{j_2}} \\ F(j_1) & & F(j_2) \end{array} \qquad (13)$$

factors through the diagram F.

8 Definition of Dependency Category

A *dependency category* is a category **C** with distinguished morphisms and a terminal object 1 and :

(i) for any object A of **C** the unique morphism $t_A : A \to 1$ is a d-morphism

(ii) for all diagrams D of d-morphisms

$$\begin{array}{ccc} B & & B' \\ {}_{x}\searrow & & \swarrow_{x'} \\ & A & \end{array}$$

an object $B_1 \underset{D}{\bowtie} B_2$ of **C** and d-morphisms $p_1 : B_1 \underset{D}{\bowtie} B_2 \multimap B_1$ and $p_1 : B_1 \underset{D}{\bowtie} B_2 \multimap B_1$ such that :

$$\begin{array}{ccc} & B_1 \underset{D}{\bowtie} B_2 & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ B_1 & & B_2 \end{array}$$

is a pullback in category **C** and such that for any cone $\langle N, \psi \rangle$ to diagram D the mediating morphism $h : N \to B_1 \underset{D}{\bowtie} B_2$ is a d-moprphism iff every morphism in the cone $\langle N, \psi \rangle$ is a d-morphism and the cone is tight.

(iii) Other pullbacks: if

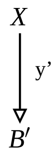
$$\begin{array}{ccc} & & X \\ & & \downarrow^{y} \\ A & \xrightarrow[f]{} & B \end{array}$$

then there is a pullback

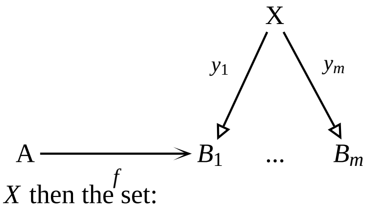
$$\begin{array}{ccc} X[f|y] & \xrightarrow{q(f,y)} & X \\ {}^{f^*y}\downarrow & & \downarrow^{y} \\ A & \xrightarrow[f]{} & B \end{array}$$

and pullbacks cohere[4] and if also:

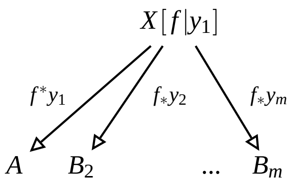
[4] exactly as for contextual category



and $y \neq y'$ then $q(f,y) \circ y'$ is a d-morphism. Further, if

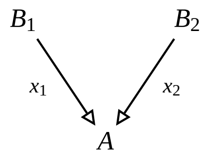


and $y_1, \ldots y_m$ is a basis for X then the set:

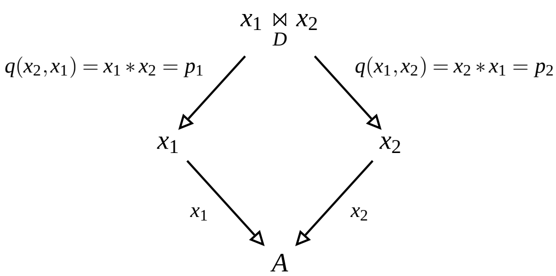


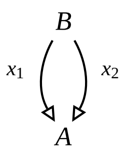
is a basis for $f^* X$.

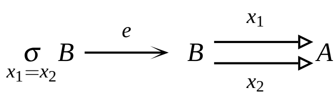
(iv) If in (iii) the morphism f is a dependency then $q(f,y)$ is also a dependency and for any diagram D:



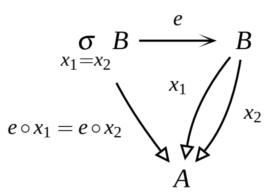
then $B_2[x_1|x_2] = B_1[x_2|x_1] = B_1 \underset{D}{\bowtie} B_2$ and $q(x_1,x_2) = x_2^* x_1$ and $q(x_2,x_1) = x_1^* x_2$ so that we have:



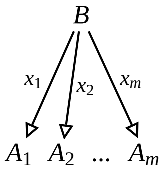
(v) Equalisers and properties of equalisers: If 

then there is an equaliser: 

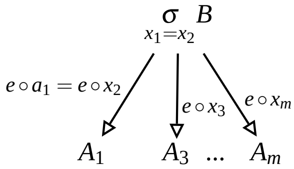
and the morphism $e \circ x_1$ which equals, by definition, $e \circ a_2$ is a d-morphism as shown in the following diagram:



For any other d-morphism $a' : B \rightarrowtriangle A'$ leaving B, the morphism $e \circ a'$ is a d-morphism; furthermore, if

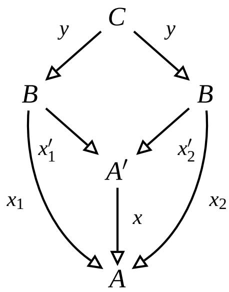


and $x_1, \ldots x_m$ is a basis for B then



is a basis for $\underset{x_1=x_2}{\sigma} B$.

For any object C and morphism $y : C \to B$ such that $y \circ x_1 = y \circ x_2$ the mediating morphism $h : C \to \underset{x_1=x_2}{\sigma} B$ is a d-morphism iff the morphism y is a d-morphism and the cone $\langle C, y \rangle$ is tight i.e iff there does not exist an object A' distinct from A and morphisms $x_1' : B \rightarrowtriangle A'$, $x_2' : B \rightarrowtriangle A'$ and $x : A' \rightarrowtriangle A$ such that each of the three diagrams contained here:

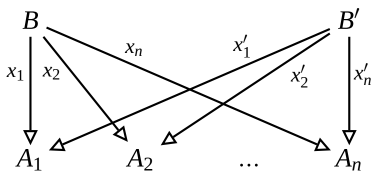


commute.

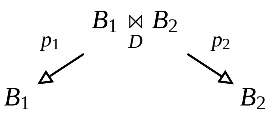
This completes the definition.

9 Construction of Limits of Other Dependency Diagrams

Lemma 9.1. *For all diagrams D of d-morphisms with shape $\bowtie_n$:*



*there is an object $B_1 \underset{D}{\bowtie} B_2$ of **C** and d-morphisms $p_1 : B_1 \underset{D}{\bowtie} B_2 \rightarrowtriangle B_1$ and $p_1 : B_1 \underset{D}{\bowtie} B_2 \rightarrowtriangle B_1$ such that :*



is a limit of diagram D in category $\mathbf{C}$ *and such that for any cone* $\langle N, \psi \rangle$ *to diagram D the mediating morphism* $h : N \to B_1 \underset{D}{\bowtie} B_2$ *is a d-morphism iff each morphism in the cone* $\langle N, \psi \rangle$ *is a d-morphism and the cone is tight.*

Proof. This lemma can be proved by induction using the next lemma. □

Lemma 9.2. *In any category* $\mathbf{C}$ *if* $F : S \to \mathbf{C}$ *is a diagram with limit* $\langle L, \phi \rangle$*, if* S' *is a category extending S by an object* β_0 *and a pair of morphisms* $\xi_1 : \beta_1 \to \beta_0$ *and* $\xi_2 : \beta_2 \to \beta_0$*, where* β_1 *and* β_2 *are objects of S, if* $G : S \to \mathbf{C}$ *is a diagram that extends F, then if :*

$$E \xrightarrow{\;e\;} L \underset{\phi_{B'} \circ G(\xi_2)}{\overset{\phi_B \circ G(\xi_1)}{\rightrightarrows}} \beta_0 \tag{14}$$

is an equaliser in $\mathbf{C}$ *then* $\langle E, \phi' \rangle$ *is a limit of the diagram G, where* ϕ' *is the cone defined by*

$$\phi'_\beta = \begin{cases} e \circ \phi_\beta & \text{if } \beta \text{ is an object of } S \\ \phi_{\beta_1} \circ G(\xi_1) & \text{if } \beta \text{ is } \beta_0 \end{cases}$$

Proof. If $\langle N, \psi' \rangle$is a cone to the diagram G then, the restriction ψ to objects in F is a cone to S. Therefore there exists a unique $h : N \to L$ such that

$$h \circ \phi = \psi \tag{15}$$

Now, we have

$$\begin{aligned} h \circ \phi_{\beta_1} \circ G(x) &= \psi_{\beta_1} \circ G(\xi_1) && \text{by (15)} \\ &= \psi_{\beta_2} \circ G(\xi_2) && \text{because } \psi \text{ is a cone to } G \\ &= h \circ \phi_{\beta_2} \circ G(\xi_2) && \text{by (15)} \end{aligned}$$

and so it follows, since (14) is an equaliser diagram, that there is a unique $g : N \to E$ such that

$$g \circ e = h \tag{16}$$

To show that $\langle E, \phi' \rangle$ is a limit to the diagram G we show that $g \circ \phi' = \psi'$ and that g is the unique such morphism, $g : N \to E$. We need show that for all objects β of S', $(g \circ \phi')_\beta = \psi'_\beta$. We need consider two cases. In the first case for objects β of S we have:

$$\begin{aligned} g \circ \phi'_\beta &= (g \circ e) \circ \phi_\beta && \text{by definition of } \phi \\ &= h \circ \phi_\beta && \text{by (16)} \\ &= \psi_\beta && \text{by (15)} \\ &= \psi'_\beta && \text{by definition of } \psi \end{aligned}$$

In the second case the object β is the additional object β_0 of S'. For this object we have:

$$\begin{aligned} g \circ \phi'_{\beta_0} &= g \circ e \circ \phi_{\beta_1} \circ G(\xi_1) && \text{by definition of } \phi' \\ &= h \circ \phi_{\beta_1} \circ G(\xi_1) && \text{by (16)} \\ &= \psi_{\beta_1} \circ G(\xi_1) && \text{by (15)} \\ &= \psi'_{\beta_1} \circ G(\xi_1) && \text{by definition of} \psi \\ &= \psi'_{\beta_0} && \text{since } \psi' \text{ is a cone} \end{aligned}$$

and so, as required, we have shown that $g \circ \phi' = \psi'$. Finally, if $g' : N \to E$ such that $g' \circ \phi' = \psi'$ then we have for any object β of S that

$$(g' \circ \phi')_\beta = \psi'_\beta$$

and therefore from the defintions of ϕ' and ψ we have that, for all objects β of S,

$$g' \circ e \circ \phi_\beta = \psi_\beta$$

and from the definition of h as the unique morphism such that $h \circ \phi = \psi$ we have that $g' \circ e = h$.

Now we have:

$$g' \circ e = h = g \circ e$$

from which it follows $g = g'$ because e is an equaliser and therefore is a monomorphism. □

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